

On a theorem of Ihara

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Abstract

Let p be a prime number and let n be a positive integer prime to p . By an Ihara-result, we mean existence of an injection with torsion-free cokernel from a full lattice in the space of p -old modular forms, into a full lattice in the space of all modular forms of level np . In this paper, we prove Ihara-results for genus two Siegel modular forms, Siegel-Jacobi forms and for Hilbert modular forms. Ihara did the genus one case of elliptic modular forms [Ih]. We propose a geometric formulation for the notion of p -old Siegel modular forms of genus two using clarifying comments by R. Schmidt [Sch] and then follow suggestions in an earlier paper [Ra1] on how to prove Ihara results. We use the main theorem in [Ra1] where we have extended an argument by G. Pappas to prove torsion-freeness of certain cokernel using the density of Hecke-orbits in the moduli space of principally polarized abelian varieties and in the Hilbert-Blumenthal moduli space which was proved by C. Chai [Ch].

Introduction

In this paper, we are interested in proving arithmetic results for modular forms by introducing geometric interpretations of some analytic aspects of the theory of modular forms. More precisely, we suggest a geometric definition for the analytic notion of old Siegel modular forms and then generalize an arithmetic result of Ihara to the case of genus two Siegel modular forms, Siegel-Jacobi forms and Hilbert modular forms.

The theorem of Ihara is used by Ribet to get congruences between elliptic modular forms. The fundamental paper [Ri1] together with his ideas in [Ri2] which were generalized by himself and many others lead to an almost complete classification of congruences between modular forms of different weights and levels in the genus one case. The result of Ihara is crucial in the procedure of raising the level.

To generalize this result to the Siegel case, one has to introduce a geometric characterization for the space of p -old Siegel modular forms. We introduce explicit correspondences, which we call Atkin-Lehner correspondences, on appropriate Siegel moduli spaces which generate the whole p -old part out of the

pull back copy of modular forms of level n inside those of level np for square-free integer n . In fact, using the Atkin-Lehner correspondences, we define a map from four copies of the space of forms of level n to the p -old part of forms of level np which turns out to be injective and generate the whole p -old part. By an Ihara-result we mean cokernel torsion-freeness of the map induced on the specified full lattices in these vector spaces. This is what Ihara proved in the elliptic modular case [Ih]. Injection of this map is an automorphic fact. But cokernel torsion-freeness is proved by getting an injection result in finite characteristic. We generalize a result of G. Pappas to get this injection using density of Hecke orbits. This density result is proved by C. Chai [Ch]. The precise statement of our main result is as follows.

Main Theorem 0.1 *Let p be a prime which does not divide the square-free integer n . Atkin-Lehner correspondences induce an injection from a full lattice in the space of p -old Siegel modular forms, into a full lattice in the space of all modular forms of level np . The cokernel of this map is free of l -torsions for all primes l not dividing $2np[\Gamma_0(p) : \Gamma'(p)]$ with $l - 1 > k$ where $\Gamma_0(p)$ and $\Gamma'(p)$ are certain congruence subgroups of $Sp(4, \mathbb{Z})$.*

This is an implication of theorems 2.2 and 2.4. Using the same ideas one can also prove an Ihara result for Siegel-Jacobi forms of genus two. Let $B_g^{0*}(n)$ denotes the compactification of the universal abelian variety over the Siegel space $A_g^0(n)$ (Find precise definitions in sections 1 and 3).

Theorem 0.2 *Let p be a prime which does not divide n . The Atkin-Lehner correspondences induce a cokernel torsion-free injection*

$$H^0(B_2^{0*}(n)/\mathbb{Z}_l, \omega^{\otimes k} \otimes L^{\otimes m})^{\oplus 4} \rightarrow H^0(B_2^{0*}(np)/\mathbb{Z}_l, \omega^{\otimes k} \otimes L^{\otimes m})$$

for all primes l not dividing $2pn[\Gamma_0(p) : \Gamma'(p)]$ with $l - 1 > k$.

Theorem 3.1 is a more precise statement. To prove an Ihara result in the case of Hilbert modular forms is much easier, since we only have two copies of Hilbert modular forms of level n inside p -old Hilbert modular forms of level np . Since our ideas work for the higher dimensional case of Hilbert-Blumenthal moduli space, we include an Ihara result for Hilbert modular forms as well. Although, one can easily prove such a result using appropriate Shimura curves.

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1 The arithmetic and geometry of Siegel spaces

In this paper, we only work with geometric formulation of Siegel modular forms. For automorphic motivations on how to define the space of p -old Siegel modular forms we refer to [Ra1].

By the Siegel upper half-space which we denote by \mathbb{H}_g we mean the set of complex symmetric g by g matrices Ω with positive-definite imaginary part. The quotient of Siegel upper half-space \mathbb{H}_g by $Sp(2g, \mathbb{Z})$ acting on \mathbb{H}_g via Möbius transformations is a complex analytic stack A_g which could be thought of as the moduli space of principally polarized abelian varieties. The universal family of abelian varieties over \mathbb{H}_g is given by $A(\Omega) = \mathbb{C}^g / (\mathbb{Z}^g \oplus \Omega \cdot \mathbb{Z}^g)$.

A Siegel modular form of weight k is a holomorphic section of the line bundle $\mathcal{O}_{\mathbb{H}_g} \otimes (\wedge^g \mathbb{C}^g)^{\otimes k}$ which we denote of ω . When written formally, this becomes an expression of the form $f(\Omega)(dz_1 \wedge \dots \wedge dz_g)^{\otimes k}$ where f is an $Sp(2g, \mathbb{Z})$ -invariant complex holomorphic function on \mathbb{H}_g which is also holomorphic at ∞ . For genus ≥ 2 the latter condition is automatically satisfied by Koecher principle.

A discrete subgroup Γ of $Sp(2g, \mathbb{Z})$ is called a congruence subgroup, if it contains $\Gamma(n)$ for some positive integer n where

$$\Gamma(n) = \{\gamma \in Sp(2g, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix} \pmod{n}\}.$$

Here is an example of a congruence subgroup

$$\Gamma_0(n) = \{\gamma \in Sp(2g, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{n}\}.$$

The quotients of \mathbb{H}_g by congruence subgroups of $Sp(2g, \mathbb{Z})$ are called Siegel spaces. They can be thought of as the moduli space of principally polarized abelian varieties equipped with certain level structure. Siegel modular forms can be considered as certain differential forms on Siegel spaces.

The space of Siegel modular forms can also be formulated in the language of schemes. Let S be a base scheme. A modular form f of weight k is a rule which assigns to each principally polarized abelian variety $(A/S, \lambda)$ a section $f(A/S, \lambda)$ of $\omega_{A/S}^{\otimes k}$ over S depending only on the isomorphism class of $(A/S, \lambda)$ commuting with arbitrary base change. Here $\omega_{A/S}$ is the top wedge of tangent bundle at origin of A over S .

To define Siegel modular forms of higher level, one should equip principally polarized abelian varieties with level structures. Let ζ_n denote an n -th root of unity where $n \geq 3$. On a principally polarized abelian scheme (A, λ) over $Spec(\mathbb{Z}[\zeta_n, 1/n])$ of relative dimension g we define a symplectic principal level- n structure to be a symplectic isomorphism $\alpha : A[n] \rightarrow (\mathbb{Z}/n\mathbb{Z})^{2g}$ where $(\mathbb{Z}/n\mathbb{Z})^{2g}$ is equipped with the standard non-degenerate skew-symmetric pairing

$$\begin{aligned} \langle, \rangle : (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} &\rightarrow \mathbb{Z}/n\mathbb{Z} \\ \langle (u, v), (z, w) \rangle &\mapsto u \cdot w^t - v \cdot z^t \end{aligned}$$

Let S be a scheme over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. The moduli scheme classifying the principally polarized abelian schemes over S together with a symplectic principal level- n structure is a scheme over S and will be denoted by $A_g(n)$. The moduli scheme $A_g(n)$ over S can be constructed from $A_g(n)$ over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$ by base change.

$Sp(2g, \mathbb{Z}/n\mathbb{Z})$ acts as a group of symmetries on $A_g(n)$ by acting on level structures. We will recognize these moduli spaces and their equivariant quotients under the action of subgroups of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ as Siegel spaces. We restrict our attention to Siegel spaces over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. $A_g(n)$ is connected and smooth over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. The condition $n \geq 3$ is to guarantee that we get a moduli scheme, instead of getting only a moduli stack. The natural morphism $A_g(n) \rightarrow A_g(m)$ where m, n are positive integers ≥ 3 and $m|n$ is a finite and etale morphism over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$.

Let $B_g(n)$ denote the universal abelian variety over $A_g(n)$. The Hodge bundle ω is defined to be the pull back via the zero section $i_0 : A_g(n) \rightarrow B_g(n)$ of the line bundle $\wedge^{\text{top}} \Omega_{B_g(n)/A_g(n)}$. The Hodge bundle is an ample invertible sheaf on $A_g(n)$ and can be naturally extended to a bundle ω on $A_g^*(n)$. We could define the minimal compactification $A_g^*(n)$ by the formula

$$A_g^*(n) = \text{proj}(\oplus_{k \geq 0} H^0(A_g^*(n), \omega^{\otimes k})).$$

The graded ring above is regarded as a $\mathbb{Z}[\zeta_n, 1/n]$ -algebra. The scheme $A_g^*(n)$ is equipped with a stratification by locally closed subschemes which are geometrically normal and flat over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. Each of these strata is canonically isomorphic to a moduli space $A_i(n)$ for some i between 0 and g . The map $A_g(n) \rightarrow A_g(m)$ can be extended uniquely to $A_g^*(n) \rightarrow A_g^*(m)$ for $m|n$. These maps when restricted to strata, induce the corresponding natural maps between lower genera Siegel spaces $A_i(n) \rightarrow A_i(m)$. The action $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ on $A_g(n)$ naturally extends to an action on the compactified Siegel space $A_g^*(n)$. This action is compatible with the maps $A_g^*(n) \rightarrow A_g^*(m)$ for $m|n$.

Let $K_0(n)$ denote the subgroup of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ fixing the g first $(\mathbb{Z}/n\mathbb{Z})$ -basis elements of $(\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$ on which $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ acts. Since $A_g^*(n)$ is a projective scheme, we can define the quotient projective schemes $A_g^{0*}(n)$ to be the geometric quotient of $A_g^*(n)$ by $K_0(n)$. This quotient provides us with a compactification of $A_g^0(n)$ which is the moduli scheme of principally polarized abelian schemes (A, λ) over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$, together with g elements in $A[n]$ generating a symplectic subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})^g$. Again we have natural maps $A_g^{0*}(n) \rightarrow A_g^{0*}(m)$ for $m|n$.

We define the Hodge bundle ω on $A_g^{0*}(n)$ to be the quotient of the Hodge bundle ω on $A_g^*(n)$ under the action of the corresponding subgroup $K_0(n)$ of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$. This is possible because the line bundle ω on the space $A_g^*(n)$ is $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ -linearizable. A Siegel modular form of weight k and full level n is a global section of ω^k on $A_g^*(n)$. Over the complex numbers, this corresponds to a Siegel modular form of weight k with respect to $\Gamma(n)$. In this paper, by a Siegel modular form of weight k and level n we mean a global section of ω^k on $A_g^{0*}(n)$. This corresponds to the congruence subgroup $\Gamma_0(n)$.

2 Ihara result for Siegel modular forms

The mod- p Bruhat decomposition implies that, we have the following decomposition

$$GSp(4, \mathbb{Q}_p) = \coprod B(\mathbb{Q}_p)w_i\Gamma_0(p),$$

where B is the Borel subgroup and w_i for $i = 1$ to 4 are running over the representatives of $W_{Sp(4, \mathbb{Z}_p)}/W_{\Gamma_0(p)}$. We choose the following representatives

$$w_1 = id, w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, w_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is suggested in [Ra1] that because of this picture, there should be 4 copies of modular forms of level n inside the p -old part which is analytically characterized by the conjugates $w_i\Gamma_0(p)w_i^{-1}$ for $i = 1$ to 4. All of these conjugates lie in $Sp(2g, \mathbb{Z}_p)$. We get the copy associated to $w_1 = id$ simply by pulling back level- n forms to level np via the projection map between moduli spaces. Following the genus one case, we can get a second copy using the action of universal p -isogeny w_p which is realized on $\Gamma_0(p)$. Now we have candidates for two of the copies of level- n forms inside the p -old part.

we are interested in finding 4 algebraic correspondences on $A_g^0(np)$ such that the image of the pull back copy under the action of these correspondences gives us all of the p -old copies. The most natural way to look for these correspondences is to pull back all the 4 copies of modular forms of level n to a congruence subgroup which is of a richer geometric structure. For example, we can pull back to $\Gamma'(n, p) = \Gamma_0(n) \cap \Gamma'(p)$ where

$$\Gamma'(p) = \{\gamma \in Sp(4, \mathbb{Z}) | \gamma \equiv \text{diag}(*, *, *, *) \pmod{p}\}.$$

The 4 specified elements of the Weyl group induce 4 involutions on the moduli space $A'_g(n, p)$ corresponding to $\Gamma'(n, p)$. Indeed, the conjugations $w_i\Gamma'(p)w_i^{-1}$ stabilize the congruence group. These involutions have a nice simple interpretation in terms of the moduli property. We could also work with $\Gamma'(np)$ and the associated Siegel space $A'_g(np)$.

The Siegel space associated to $\Gamma'(p)$ is the quotient of the level- p Siegel space by the subgroup of $Sp(2g, \mathbb{Z}/p\mathbb{Z})$ which stabilizes all of the $2g$ copies of $\mathbb{Z}/p\mathbb{Z}$ in $(\mathbb{Z}/p\mathbb{Z})^{2g}$. The conjugations correspond to symplectic automorphisms which are well defined on the kind of level structure we are considering here. It is essential to note that we can not obtain the 4 copies of p -old modular forms by applying the above 4 involutions on the direct pull back of modular forms via the natural map $\pi : A'_g(n, p) \rightarrow A_g(n)$. Because forms in the pull-back are already invariant under all w_i 's. Instead, we shall apply w_p after pulling these forms back as far as $A_g^0(np)$ and then pull them back to $A'_g(n, p)$. The second copy we get in this manner, generates a new copy of p -old forms on $A'_g(n, p)$

by applying involutions by Weyl elements, to the pull back of the second copy generated by w_p which we can push forward down to $A_g^0(np)$. The forth copy can be constructed by applying w_p again to the final copy. We propose the following definition for the space of p -old Siegel forms

Definition 2.1 *The space of p -old Siegel modular forms of level np is generated by the images of correspondences $id, w_p, \pi'_* \circ w_2 \circ \pi'^* \circ w_p, w_p \circ \pi'_* \circ w_2 \circ \pi'^* \circ w_p$ acting on the pull-back copy of Siegel modular forms of level n inside those of level np . These are called the Atkin-Lehner correspondences. The space of new forms is defined to be the orthogonal complement of the space of p -old forms with respect to the Petterson inner product.*

The careful considerations of R. Schmidt shows that this is a well-defined notion of old-form for square-free n (Look at table one in [Sch]). Moreover, one can prove that these correspondences produce all of the 4 copies we are expecting inside the space of Siegel modular forms of level np . Here is the precise statement

Theorem 2.2 *Let p be a prime which does not divide n . The correspondences $id, w_p, \pi'_* \circ w_2 \circ \pi'^* \circ w_p$ and $w_p \circ \pi'_* \circ w_2 \circ \pi'^* \circ w_p$ induce a cokernel torsion-free injection*

$$H^0(A_2^{0*}(n)/\mathbb{Z}_l, \omega^{\otimes k})^{\oplus 4} \rightarrow H^0(A_2^{0*}(np)/\mathbb{Z}_l, \omega^{\otimes k})$$

for all primes l not dividing $2np[\Gamma_0(p) : \Gamma'(p)]$ with $l - 1 > k$.

This theorem is the main theorem in [Ra1] which is obtained by generalizing the following result of G. Pappas [Pa]. One shall note that arguments in [Ra1] work only for genus 2. The general case is treated in the new version [Ra2].

Theorem 2.3 (Pappas) *Let F be a field of characteristic zero or finite characteristic q with q not dividing pn . Let f and g be modular forms on $A_g^0(pn)/F$ which are pulled back from $A_g^0(n)$. Then $w_p \cdot f + g = 0$ implies $f = g = 0$ except when $p-1|k$ and the characteristic is finite. In this case, the kernel is generated by $(H^m, -H^m)$ where H is the Hasse invariant.*

For interested reader, we will reproduce the result of G. Pappas with a few modifications and simplified proofs in the appendix.

Theorem 2.4 *Let n be a square-free integer. The space of p -old Siegel modular forms is generated by Siegel modular forms of level np which are eigenforms of Hecke operators and appear as Siegel eigenforms with the same eigenvalues in the space of Siegel forms of level n .*

Proof. There could be at most four copies of Siegel modular forms of level n inside p -old Siegel modular forms of level np . The four correspondences commute with all Hecke operators T_q with q prime to p . Therefore, by corollary 5.3 of [Sch] they produce old forms. By theorem 2.2 the four images are linearly independent. Therefore, every p -old Siegel modular form is generated by the images of these correspondences. \square

Now, it is clear that our main theorem follows from theorems 2.2. and 2.4.

3 Ihara result for Siegel-Jacobi modular forms

The universal abelian variety $B_g(n)$ hosts the line bundle $\wedge^* \Omega_{B_g(n)/A_g(n)}$ which can be naturally extended to a bundle on the universal semi-abelian variety $A_g^*(n)$. We denote this line bundle by $\pi^* \omega$ where π is the projection from the universal abelian variety to the base. Unlike the Hodge bundle $\pi^* \omega$ is not an ample invertible sheaf on $B_g(n)$.

One can construct an ample line bundle on the universal abelian variety $B_{g,n}$ via the principal polarization of $B_g(n)$. Let

$$\Delta : B_g(n) \rightarrow B_g(n) \times_{A_g(n)} B_g(n)$$

denote the diagonal map. We set $L = \Delta^* P_{B_g(n)}$ where P is chosen to be the invertible sheaf which is twice the principal polarization of $B_g(n)$. The invertible sheaf $\pi^* \omega^{\otimes k} \otimes L^{\otimes m}$ is ample on $B_g(n)/\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$ for $k, m \gg 0$.

Faltings and Chai construct an arithmetic compactification of the fiber power of universal abelian variety over the base (theorem 1.1 of chapter VI in [Fa-Ch]). Let $Y = B_g(n)_{A_g(n)}^{\times s}$ and $\pi : Y \rightarrow A_g(n)$ denote the natural projection to the base and let $T_G = \text{Lie}(G_{g,n})^s$. Then there exists an open embedding $Y \rightarrow \hat{Y}$ and a proper extension of the projection over the base $\hat{\pi} : \hat{Y} \rightarrow \hat{A}_g(n)$ such that i) \hat{Y} and $\hat{A}_g(n)$ are smooth over $\mathbb{Z}[1/n, \zeta_n]$ and the complement $\hat{Y} - Y$ is a relative divisor with normal crossings.

ii) The translation action of Y on itself extends to an action of $G_{g,n}^s$ on \hat{Y} .

iii) $\hat{\Omega}_{Y/A_g(n)}^1 = \hat{\Omega}_Y^1 / \hat{\pi}^*(\hat{\Omega}_{A_g(n)}^1)$ is locally free and isomorphic to $\hat{\pi}^*(T_{G_{g,n}^{*s}})$.

iv) $R^a \hat{\pi}_*(\wedge^b \hat{\Omega}_{Y/A_g(n)}^*) = (\wedge^a T_{G_{g,n}}^s) \otimes (\wedge^b T_{G_{g,n}}^{*s})$.

where all these isomorphisms extend the canonical isomorphisms over $A_g(n)$.

Here $\hat{\Omega}_{Y/S}^1$ is an abbreviation for $\Omega_{Y/S}^1[d\log \infty]$. In case $s = 1$ we denote this compactification by $B_g^*(n)$. The line bundles $\pi^* \omega$ and L naturally extend over this compactification. The compactified universal abelian variety $B_g^*(n)$ does not contain the universal semi-abelian variety over the base. This compactification is $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ -linearizable and one gets compactifications $B_g^{0*}(n)$ over $A_g^{0*}(n)$.

Let $m, n, k \in \mathbb{N}$ and let R be a $\mathbb{Z}[1/n, \zeta_n]$ -module. A Siegel-Jacobi form f of genus g , weight k , index m and full level n with coefficients in R is an element in

$$J_{k,m}(n)(R) = H^0(B_g(n)/R, \pi^* \omega^{\otimes k} \otimes L^{\otimes m}).$$

The action of double-cosets as geometric correspondences does not preserve $J_{k,m}(n)$. However the action of the Hecke algebra $H_p^{(2)}$ associated to the pair

$$(GSp^{(2)}(2g, \mathbb{Q}_p), GSp^{(2)}(2g, \mathbb{Z}_p))$$

can be defined [Kr]. One can show that this Hecke algebra is isomorphic to the algebra generated by connected components of p^2 -isogenies. By p^2 -isogenies, we mean isogenies of over $\mathbb{Z}[1/p]$ between two principally polarized abelian scheme

over $\mathbb{Z}[1/p]$ which are of degree P^{2eg} where e is a positive integer. Each connected component of p^2 -isogenies acts on $J_{k,m}(n)$ via the following diagram

$$\begin{array}{ccccc}
 & \pi_1^* B_{g,n} & \xrightarrow{\pi_0} & \pi_2^* B_{g,n} & \\
 \pi'_1 \swarrow & & \searrow & & \searrow \pi'_2 \\
 B_{g,n} & & Z & & B_{g,n} \\
 \searrow & & \swarrow \pi_1 & \searrow \pi_2 & \swarrow \\
 & A_g(n) & \xrightarrow{id} & A_g(n) &
 \end{array} \tag{1}$$

where π_0 is the universal isogeny over Z and the maps π_1 and π_2 are natural projections from the appropriate Siegel space Z . If ϕ is an isogeny of degree p^{2eg} , then we have an isomorphism $\phi^* \pi_2^* L \cong \pi_1^* L^{\otimes p^{2e}}$. We define the action of Z on $J_{k,m}(n)$ in the following manner

$$\begin{aligned}
 J_{k,m}(n) &\xrightarrow{\pi'_2} H^0(\pi_2^* B_{g,n}, \pi_2^*(\omega^{\otimes k} \otimes L^{\otimes m})) \xrightarrow{\Phi^*} H^0(\pi_1^* B_{g,n}, \pi_1^*(\omega^{\otimes k} \otimes L^{\otimes p^{2e}.m})) \\
 &\xrightarrow{\pi'_1} H^0(B_{g,n}, \omega^{\otimes k} \otimes L^{\otimes p^{2e}.m}) \cong H^0(B_{g,n}, \omega^{\otimes k} \otimes [P^2]^* L^{\otimes m}) \rightarrow J_{k,m}(n)
 \end{aligned}$$

The last mapping is possible to be defined because of finiteness and flatness of multiplication by p .

In order to get appropriate correspondences on the universal abelian variety $B_g^{0*}(n)$ we construct compactifications $B'_g(n, p)$ of the universal abelian variety $B_g(n, p)$ over the Siegel spaces $A'_g(n, p)$. In the particular case of genus two, the involutions w_i for $i = 1$ to 4 on $A'_2(n, p)$ coming from the Weyl elements, induce 4 involutions on the compactified universal abelian varieties which will be denoted again by the same notation $w_i : B_2^{0*}(n, p) \rightarrow B_2^{0*}(n, p)$. The universal p^2 -isogeny on $A_2^{0*}(n)$ can be extended to a map $w_p : B_2^{0*}(np) \rightarrow B_2^{0*}(np)$. We extend the projection $\pi' : A'_g(n, p) \rightarrow A_g^0(np)$ to $\pi' : B_g^0(np) \rightarrow B_g^0(np)$.

Theorem 3.1 *Let p be a prime which does not divide n . The geometric correspondences $id, w_p, \pi'_* \circ w_2 \circ \pi'^* \circ w_p$ and $w_p \circ \pi'_* \circ w_2 \circ \pi'^* \circ w_p$ induce a cokernel torsion-free injection*

$$H^0(B_2^{0*}(n)/\mathbb{Z}_l, \omega^{\otimes k} \otimes L^{\otimes m})^{\oplus 4} \rightarrow H^0(B_2^{0*}(np)/\mathbb{Z}_l, \omega^{\otimes k} \otimes L^{\otimes m})$$

for all primes l not dividing $2pn[\Gamma_0(p) : \Gamma'(p)]$ with $l - 1 > k$.

Proof. One can restrict Siegel-Jacobi forms to the base in order to get Siegel modular forms. This geometric restriction commutes with the action of correspondences and the map between cohomology groups. The restricted map with coefficients in a field F

$$H^0(A_2^{0*}(n)/F, \omega^{\otimes k} \otimes L^{\otimes m})^{\oplus 4} \rightarrow H^0(A_2^{0*}(np)/F, \omega^{\otimes k} \otimes L^{\otimes m})$$

is proved to be injection for F of characteristic zero or characteristic l for l not dividing $2pn[\Gamma_0(p) : \Gamma'(p)]$ with $l - 1 > k$. This implies injectivity of the map between Siegel-Jacobi forms with coefficients in F , because by applying multiplication by l^n one can get Siegel-Jacobi forms of higher index restricting to a nonzero Siegel modular form on the base. This implies cokernel torsion-freeness of the map between Siegel-Jacobi forms. \square

4 Ihara result for Hilbert modular forms

Here, we consider Hilbert modular forms in geometric framework (look at [Wi] for both geometric and analytic approaches). Fix a totally real number field F with $g = [F : \mathbb{Q}] > 1$. Let $O = O_F$ denote the ring of integers of F and D_F its different. Also, fix a basis a_1, \dots, a_g for O_F over \mathbb{Z} and thus an isomorphism

$$O_F/nO_F \rightarrow (\mathbb{Z}/n\mathbb{Z})^g.$$

We are interested in pairs $(X/s, m)$ consisting of an abelian scheme X/S over S of relative dimension g and a homomorphism $m : O_F \rightarrow \text{End}(X/S)$ such that the relative Lie algebra $\text{Lie}(X/S)$ is locally a free $O_F \otimes_{\mathbb{Z}} O_S$ -module of rank one on the Zariski site. Let $c^\sharp = (c, c_+)$ be a projective rank-1 O_F -module equipped with a notion of positivity, i.e. an ordering of the rank one free \mathbb{R} -module $c \otimes_{\tau, O_F} \mathbb{R}$ for each embedding $\tau : F \hookrightarrow \mathbb{R}$. A c^\sharp -polarization is an isomorphism of étale sheaves of O_F -modules on S with positivity

$$\lambda : (P, P_+) \xrightarrow{\sim} (c, c_+)$$

where $P = P(X, m)$ is the sheaf $T/S \mapsto \text{Hom}_{T, O}(X_T, X_T^t)_{\text{sym}}$ of O -linear quasi-polarizations of X with P_+ the subsheaf of O -linear polarizations of X .

A principal level- n structure on the pair $(X/s, m)$ is an O -linear symplectic isomorphism

$$\alpha : X[n] \rightarrow (O_F/nO_F)^2 = (\mathbb{Z}/n\mathbb{Z})^{2g},$$

where $X[n] \subset X$ is the subgroup scheme of the torsion points of X . α induces an isomorphism

$$P(X, m) \otimes_{O_F} \wedge_{O_F}^2 X[n] \xrightarrow{\sim} D_F^{-1}/nD_F^{-1}(1).$$

Let n be an integer ≥ 3 . Let S be a $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$ -scheme. The moduli scheme over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$ which classifies c^\sharp -polarized pairs (X, m) over S which are equipped with a principal level- n structure is denoted by $M(c^\sharp, n)$. This is a smooth and faithfully flat scheme of relative dimension g . Let $m|n$ be an integer ≥ 3 . The natural morphism

$$M(c^\sharp, n) \rightarrow M(c^\sharp, m)$$

is finite and flat over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. The moduli scheme $M(c^\sharp, n)$ depends on the isomorphism class of c^\sharp in the ideal class group $Cl(F)$ of O_F .

The moduli space with polarization varying in the class group is the full Hilbert-Blumenthal moduli scheme $M(\text{pol}, n)$ with level- n structure. Like the Siegel case, there exists a universal abelian variety over $M(c^\sharp, n)$. We define the Hodge bundle ω as the top wedge of the relative dualizing sheaf. The Hodge bundle is an ample line bundle. Koecher principle will still remain valid. Namely sections of $\omega^{\otimes k}$ over $M(c^\sharp, n)$ naturally extend to honest sections on the compactification $M^*(c^\sharp, n)$. Indeed, we have a natural compactification

$$M^*(c^\sharp, n) = \text{Proj} \left(\sum_{k \geq 0} H^0(M(c^\sharp, n), \omega^{\otimes k}) \right)$$

where the graded ring is considered as an algebra over $\mathbb{Z}[\zeta_n, 1/n]$. This is a canonical projective scheme $M^*(c^\sharp, n)$ of finite type over $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$, containing $M(c^\sharp, n)$ as a dense open subscheme. The complement is finite and flat. In fact this complement is isomorphic to a disjoint union of finitely many copies of $\text{Spec}(\mathbb{Z}[\zeta_n, 1/n])$. A cusp of $M(c^\sharp, n)$ is characterized by an extension of projective O_F -modules

$$0 \rightarrow D_F^{-1}a^{-1} \rightarrow K \rightarrow b \rightarrow 0$$

and an O_F -linear isomorphism

$$\gamma : K/nK \xrightarrow{\sim} (O_F/nO_F)^2$$

where a and b are rank-1 projective modules equipped with an isomorphism

$$\beta : b^{-1}a \xrightarrow{\sim} c.$$

The number of cusps of $M(c^\sharp, 1)$ is equal to the class number of O_F . From this data, for every cusp of $M(c^\sharp, n)$ we get an isomorphism

$$c \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} D_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

induced by β .

The group $Sp(2, O_F/nO_F)$ acts naturally on $M^*(c^\sharp, n)$ by acting on the level structure at each point. This action is compatible with the morphism

$$M(c^\sharp, n) \rightarrow M(c^\sharp, m)$$

for $m|n$. We can consider quotients of the moduli scheme by subgroups of $Sp(2, O_F/nO_F)$. We can define a compactification $M^{0*}(c^\sharp, n)$ for $M^0(c^\sharp, n)$ by normalization of

$$M^0(c^\sharp, n) \rightarrow M^*(c^\sharp, n).$$

Let R be a $\mathbb{Z}[\zeta_n, 1/n]$ -algebra. We define a Hilbert modular form of genus g and weight k , with coefficients in R regular at cusps, to be an element of $H^0(M(pol, n), \omega^{\otimes k} \otimes R)$. Koecher principle is also valid for Hilbert modular forms. In other words, any Hilbert modular form with coefficients in R extends to a unique section of $\omega^{\otimes k}$ on $M^*(pol, n)$. From the moduli property, one can define an embedding of $M^*(pol, n)$, the compactified moduli stack of polarized pairs (X, m) into A_g^* taking cusps to cusps, such that ω on A_g^* restricts to ω on $M^*(pol, 1)$. This way, one can get Hilbert modular forms by restriction of Siegel modular forms to the Hilbert-Blumenthal moduli space [Wi].

Dividing by subgroups of $Sp(2, O_F/nO_F)$ acting on $M_g^*(n)$ one gets etale quotients of Hilbert-Blumenthal moduli space of level n . Consider the congruence subgroup of $SL(2, O_F)$ defined by

$$\Gamma^0(n) = \{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F) | a, d \in O_F, b \in \delta^{-1}, c \in n\delta, ad - bc \in O_F^\times\}.$$

Note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma^0(p) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1}\delta^{-2} \end{pmatrix} \Gamma^0(p) \begin{pmatrix} 1 & 0 \\ 0 & p^{-1}\delta^{-2} \end{pmatrix}^{-1}.$$

Thus, we have an involution w_p on the moduli space $M^0(pol, pn)$ associated to $\Gamma^0(pn)$ whose level structure is defined by fixing a symplectic subgroup of abelian scheme of np -torsion points $(O_F/npO_F)^2$ isomorphic to the group scheme (O_F/pO_F) . Having w_p we get a map

$$\pi^* \oplus w_p \circ \pi^* : H^0(M^0(pol, n), \omega^{\otimes k}/O_F)^{\oplus 2} \rightarrow H^0(M^0(pol, pn), \omega^{\otimes k}/O_F)$$

Theorem 4.1 *Let p be a prime ideal which does not divide n . There exists a cokernel torsion-free injection*

$$H^0(M^0(pol, n)/O_{F,\ell}, \omega^{\otimes k})^{\oplus 2} \rightarrow H^0(M^0(pol, pn)/O_{F,\ell}, \omega^{\otimes k})$$

for all primes l of the ring O_F not dividing $2pn$ with $l - 1 > k$. Here $O_{F,\ell}$ denotes the ℓ -adic localization of O_F and l is the characteristic of its residue field.

The proof is in the same lines as the appendix. But, one can easily prove such a result using appropriate Shimura curves. This is why we leave the proof to interested reader.

Appendix

We shall thank G. Pappas, who communicated his results to us before publication. His results works both in finite characteristics and characteristic zero. The idea is essentially due to F. Diamond and R. Taylor [Di-Ta]. Pappas uses density of Hecke orbits to apply the ideas of [Di-Ta] in a much more general setting. Density of Hecke orbits was proved by C. Chai [Ch].

Many results in this section do not hold, when $g = 1$. We assume $g \geq 2$ through the whole section. Let F be a field of characteristic 0 or of finite characteristic q , not dividing the level np .

Proposition 4.2 *The line bundle $\omega^{\otimes k}$ on $A_g^0(n)$ is nontrivial for nonzero k .*

Proof. Moret-Bailly and Oort introduce a principally polarized abelian scheme $A' \rightarrow \mathbb{P}^1(F)$ of relative dimension two with restriction of ω on $\mathbb{P}^1(F)$ isomorphic to $O(q-1)$. The local system of n -torsion points is trivial over $\mathbb{P}^1(F)$ thus by adding a constant family of right dimension to A' we can induce an embedding $\mathbb{P}^1(F) \rightarrow A_g^0(n)$. This implies the claim. \square

Remark 4.3 . *On $X^0(3)$ we have $\omega = O_{\mathbb{P}^1(F)}(1)$ and on $X^0(4)$ we have $\omega = O_{\mathbb{P}^1(F)}(3)$. In both cases we can have a divisor for ω supported in the cusps. \square*

Let S be a scheme of characteristic q , and $A \rightarrow S$ be a semi-abelian scheme over S . Frobenius morphism induces a homomorphism

$$Fr^* : e^* \Omega_{A/S}^1 \rightarrow (e^* \Omega_{A/S}^1)^{(q)}$$

where $(e^* \Omega_{A/S}^1)^{(q)}$ is the same as $e^* \Omega_{A/S}^1$ except that, the underlying O_S -module structure is combined with Frobenius. Therefore, we get a homomorphism $\det(Fr^*) : \omega_{A/S} \rightarrow \omega_{A/S}^{\otimes q}$. Since $\det(e^* \Omega_{A/S}^1)^{(q)} = \det(e^* \Omega_{A/S}^1)^{\otimes q}$, over $\mathbb{Z}/q\mathbb{Z}$ this homomorphism corresponds to a section H of $\omega^{\otimes(q-1)}$.

Theorem 4.4 *Let $\pi : A_g^0(np) \rightarrow A_g^0(n)$ be the projection. Then the morphism*

$$\pi^* \oplus w_p \circ \pi^* : H^0(A_g^0(n)/F, \omega^{\otimes k}) \oplus H^0(A_g^0(n)/F, \omega^{\otimes k}) \rightarrow H^0(A_g^0(np)/F, \omega^{\otimes k})$$

is injective if F is of characteristic zero, or if k is not divisible by $q-1$. In case $k \equiv 0 \pmod{q-1}$ the kernel is generated by (H^m, H^{-m}) where H is the Hasse invariant and $m=k/(q-1)$.

Proof. Injection in characteristic zero can be reduced to \mathbb{C} , and in characteristic q can be reduced to $\overline{\mathbb{F}}_q$ by flat base change. Suppose (f_1, f_2) be an element in the kernel of $\pi^* \oplus w_p \circ \pi^*$. Consider an ordinary point on $A_g^0(np)/F$. It corresponds to an isogeny $\phi : A \rightarrow B$ between principally polarized abelian varieties. Let ω_A and ω_B , be a local bases for $\omega_{A/F}$ and $\omega_{B/F}$, respectively. We shall regard f_1 and f_2 as forms on $A_g^0(np)$ which are pulled back from $A_g^0(n)$. But when we address the values of them, we shall not specify the p -isogeny associated to the abelian variety on that point, because these forms being pull backs are independent of these isogenies. Then by definition of w_p , we have $f_1(A, \phi^* \omega_B) = -f_2(B, \omega_B)$, and also $f_1(B, \phi^{t*} \omega_A) = -f_2(A, \omega_A)$ for the dual isogeny $\phi^t : B \rightarrow A$. If $\psi : B \rightarrow C$ be another isogeny of the same type, we have $f_1(B, \psi^* \omega_C) = -f_2(C, \omega_C)$. Therefore, $\lambda f_2(A, \omega_A) = \mu f_2(C, \omega_C)$ for non-zero constants λ and μ . We conclude that for a chain of isogenies $A \rightarrow B \rightarrow C$ of degree P^g corresponding to points in $A_g^0(np)$, if f_2 vanishes at a point in $A_g^0(np)$ over $A \in A_g^0(n)$, then it also vanishes at every point in $A_g^0(np)$ which is over $C \in A_g^0(n)$. Let Z denote the subscheme of $A_g^0(n)$ defined by $f_2 = 0$. We shall show in the following lemmas that $Z = A_g^0(n)$. This will prove that $f_1 = f_2 = 0$ and we get injection.

In case f_2 only vanishes at supersingular points, there will be a positive integer m such that, f_2/H^m is a global non-zero section of $\omega^{m-k(q-1)}$ on $A_g^0(n)$. This implies triviality of the bundle $\omega^{m-k(q-1)}$ and thus $k = m(q-1)$. Let $\overline{S}_g(n)$ denote the divisor of H in $\overline{A}_g^0(n)$ where $\overline{A}_g^0(n)$ denotes a smooth toroidal compactification of $A_g^0(n)$ (Chapter IV in [Fa-Ch]) and let $S_g(n)$ denote the divisor of H in $A_g^0(n)$. $\overline{S}_g(n)$ is the reduced Zariski closure of $S_g(n)$. which is of pure codimension one in $\overline{A}_g^0(n)$, by construction of toroidal compactifications. The complement $A_g^0(n) - S_g(n)$ is the ordinary locus. So in this situation, we get $f_2 = \lambda \cdot H^m$ for some $\lambda \in F$. \square

Lemma 4.5 *The subscheme Z of $A_g^0(n)$ satisfies the following condition:
 (*) If $\phi : A \rightarrow B$ is a principally polarized p -isogeny over F of degree p^c with $2g|c$ and one of the points corresponding to A or B belong to $Z(F)$, then so does the other point.*

Proof. [Sh] proves that for a principally polarized isogeny $\phi : A \rightarrow B$ of degree p^{cg} over F there exists a chain of principally polarized isogenies of degree p^g whose composition equals to ϕ . The details are left to the reader. \square

Lemma 4.6 *There exists a closed subscheme Z' of $A_g^0(n)$ which contains Z and satisfies the following property
 (**) If $\phi : A \rightarrow B$ is a principally polarized p -isogeny over F and one of the points corresponding to A or B belong to $Z'(F)$ then so does the other point.*

Remark 4.7 . By results of [Ch] any closed subscheme of $A_g^0(n)$ satisfying the above property is dense in $A_g^0(n)$. \square

Proof. We are considering the case that Z has an ordinary point. Suppose Z be of codimension one. Let Z'' denote the Zariski closure of the set of all ordinary points which are connected to a point on Z by an isogeny of degree dividing p^{2g} . For every point in Z there are at most finitely many points connected to it by isogenies. Thus Z'' has also codimension one in $A_g^0(n)$. Now define $Z' = Z'' \cup S_g(n)$. Z' satisfies the property mentioned above. \square

Lemma 4.8 *Let f and g be forms as above of the same weight. To show that $w_p.f + g = 0$ implies $f = g = 0$ it is enough to prove $f = w_p.f$ implies $f = 0$ for arbitrary weight.*

Proof. This is because w_p is an involution. $w_p.f + g = 0$ implies that $(f + g)^{\otimes 2}$ and $(f - g)^{\otimes 2}$ are both w_p -invariant. If being w_p -invariant implies vanishing, we get $(f + g)^{\otimes 2} = (f - g)^{\otimes 2} = 0$ and thus $f + g = f - g = 0$ which implies vanishing of f and g . \square

Proof(2.3). Let h be a form on $A_g^0(n)$ whose pull back to $A_g^0(pn)$ is w_p -invariant. Note that if $A \rightarrow B$ is an isotropic isogeny of degree p^g between two ordinary principally polarized abelian varieties then if h vanishes at the point corresponding to A it also vanishes at the point corresponding to B . So by the argument of Pappas, the subscheme Z defined by $h = 0$ contains the full Hecke orbit of each of its ordinary points. By a result of [Ch] we know that each Hecke orbit is dense. Hence $Z = A_g$ and $h = 0$. Except if Z does not contain any ordinary point. In this case we repeat the argument of Pappas and deduce that $h = \lambda.H^m$ and $p - 1|k$. \square

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